Math 522 Exam 4 Solutions

1. For every integer n, prove that n^5 and n have the same last digit.
   BONUS: Prove that n^{10} and $2n^6 - n^2$ have the same last two digits.

   Note that the last digit of a natural number is its remainder upon division by 10; two naturals x, y share the same last digit exactly when $10|(x-y)$. [Similarly, x, y share the same last two digits exactly when their remainders upon division by 100 are the same, which is exactly when $100|(x-y)$.]

   We first prove both $2|(n^5-n)$ and $5|(n^5-n)$. If n is even, n^5-n is the difference of two even numbers and is even; if n is odd then n^5 is odd, hence n^5-n is the difference of two odd numbers and is even. Thus $2|(n^5-n)$. By Fermat's little theorem, $5|(n^5-n)$. Hence $n^5-n=2a=5b$ for some integers a, b. So $2|5b$, and $2 \nmid 5$, so $2|b$ since 2 is prime. So $b=2c$, and $n^5-n=5b=10c$, so $10|(n^5-n)$.

   BONUS:
   $n^5-n=10k$, so $n^{10}-2n^6+n^2=(n^5-n)^2=100k^2$, so $100|n^{10}-2n^6+n^2$.

2. For every prime p, prove that $p|(p-2)!-1$.

   Solution 1: This is an intermediate step in the second in-class proof of Wilson's theorem. Among $\{1, 2, \ldots, p-1\}$ most of the elements pair up as (a, b) where $a \neq b$ and $p|ab-1$. The only exceptions are 1 and $p-1$. Hence we rearrange $(p-2)! = 1 \cdot (a_1b_1)(a_2b_2)\cdots(a_kb_k)$. We now recall the lemma that if $p|x-1$ and $p|y-1$, then $p|xy-1$. [Proved in class, or via $p|(x-1)(y-1) = xy-x-y+1 = (xy-1)-(x-1)-(y-1)$]. Applying the lemma repeatedly we get $p|1 \cdot (a_1b_1)(a_2b_2)\cdots(a_kb_k)-1 = (p-2)!-1$.

   Solution 2: Working backwards from Wilson's theorem, $p|(p-1)!+1 = (p-1)(p-2)!+1 = p(p-2)!-(p-2)!+1$. But $p|p(p-2)!$, so we subtract the first from the second to get $p|p(p-2)!-\big(p(p-2)!-(p-2)!+1\big) = (p-2)!-1$.

   Solution 3 (Britte, Elizabeth): We start the same way, with Wilson's theorem, $p|(p-1)!+1$. But also $p|p$, so we subtract the second from the first to get $p|(p-1)(p-2)!+1-p = (p-1)\big((p-2)!-1\big)$. Since p is prime, either $p|p-1$ (impossible) or $p|(p-2)!-1$.

3. High score=100, Median score=73.5, Low score=52